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ENGEL GROUPS I

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ABSTRACT
In this series of papers we study Engel groups through their presentation, using Small Cancellation Theory. In the present paper we describe the consequences of a single Engel relator.

Introduction

An n-Engel relator R is an n-time iterated commutator of an element A of the free group $\mathscr{F}$ by an element $B \in \mathscr{F} : R = [A, nB]$, where $[A, 0B] = A$, $[A, nB] = [[A, (n-1)B], B]$. An n-Engel group is a group in which every pair of elements satisfies an n-Engel relator. Obviously, 1-Engel groups are just the abelian groups. 2-Engel groups were considered by F. W. Levi [7]. He showed that 2-Engel groups are nilpotent. About twenty years later, Heineken showed in [3] that 3-Engel groups without elements of order 5 are nilpotent. Several authors studied Engel groups around that time, and it was shown that Engel groups which satisfy some additional condition are locally nilpotent (see [2], [9] and [11]). Recently Zelmanov [12] proved that Lie algebras over a field with characteristic 0 satisfying an n-Engel condition are nilpotent. This strong result has consequences on Engel groups.

The aim of this series of papers is to study relatively free n-Engel groups through their presentations and, in particular, to investigate the question as to whether these groups are locally nilpotent.

In the present paper we give an explicit description of all the consequences of a single Engel relator. This is an important first step toward the study of these groups. Our basic tool is van Kampen diagrams. The paper is organized

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as follows: In Chapter 1 we deal with the maximal pieces in van Kampen diagrams. In Section 1 we describe explicitly a single Engel relator in the generators and prove some of its combinatorial properties. We then extend this result in Section 2 to arbitrary words instead of the generators. In Section 3 we determine all the maximal pieces that may occur in a van Kampen diagram coming from an n-Engel relator on two generators.

In Sections 4 and 5 we do the same for arbitrary words. At this point it turns out that diagrams coming from a single Engel relator satisfy none of the usual small cancellation conditions.

We overcome this difficulty by a construction due to E. Rips [10] — the derived diagrams. These are the subject matter of Chapter 2. After introducing some preparatory results in Section 1, we formulate our main result in Section 2. The rest of the paper is devoted to its proof.

The results of the present paper are the building blocks of the subsequent papers [5] and [6]. In [5] we solve the word problem for groups defined by n-Engel relators R_i such that $|R_i| \leqq c|R_j|$, $c = [n/2]$. Relying on this result we show in [6] that these groups are not locally nilpotent.

1. The maximal pieces

1.1. We begin with the description of the structure of our relator. First we determine it in a very special case which, however, explains the general case.

1.1.1. Let $F = \langle x, y \rangle$ be the free group generated by x and y. Let $[x, y] = x^{-1}y^{-1}xy$, $[x, 1y] = [x, y]$ and for every $i \geqq 2$ define $[x, iy] = [[x, (i-1)y], y]$. Let $w_0 = x$ and for $i \geqq 1$ define $w_i = w_{i-1}^{-1}y^{-1}w_{i-1}$. Denote $c_i = [x, iy]$. Then $c_1 = w_1y$, $c_2 = [w_1y, y] = y^{-1}w_1^{-1}y^{-1}w_1yy = y^{-1}w_2y^2$ and by induction $c_i = y^{-(i-1)}w_iy^i$. We see that c_i is not cyclically reduced, hence we prefer to write $c_i = y^{-(i-1)}(w_iy)y^{i-1}$. Since w_iy *is* cyclically reduced, we shall deal mainly with w_iy. Our first task is to determine the structure of w_i.

Define function $\delta : \mathbf{N} \to \{1, -1\} \subseteq \mathbf{Z}$ by

$$\delta(i) = \begin{cases} 1 & \text{if } i = 2^{\nu(i)}(4t+1), \\ -1 & \text{if } i = 2^{\nu(i)}(4t+3). \end{cases} \tag{0}$$

Further, let $W_0(X, Y) = X$ and for every natural number k and words $X, Y \in F$ define words

$$W_k(X, Y) = (X^{-\delta(1)}Y^{\delta(2)}X^{-\delta(3)}Y^{\delta(4)}X^{-\delta(5)}\cdots X^{-\delta(2^k-1)})Y^{-1}$$
$$\times (X^{-\delta(2^k+1)}Y^{\delta(2^k+2)}\cdots X^{-\delta(2^{k+1}-1)}).$$

Thus,

$$W_0(X, Y) = X, \qquad W_1(X, Y) = X^{-1}Y^{-1}X = W_0(X, Y)^{-1}Y^{-1}W_0(X, Y),$$
$$W_2(X, Y) = X^{-1}YXY^{-1}X^{-1}Y^{-1}X = W_1^{-1}(X, Y)Y^{-1}W_1(X, Y).$$

Observe that $W_1^{-1}(X, Y)Y^{-1}W_1(X, Y) = W_1(W_1(X, Y), Y)$. An easy induction argument shows that more generally, for any $i, j \geqq 1$, we have

$$W_i(W_j(X, Y), Y) = W_{i+j}(X, Y).$$

We summarise this, adding some further consequences of the definitions in the following proposition.

PROPOSITION. *Assume the above notation. Then*

(a) $W_n(X, Y)$ *is defined by*

$$W_0(X, Y) = X, \quad W_1(X, Y) = Y^{-X} = X^{-1}Y^{-1}X \quad \textit{and}$$
$$W_{i+j}(X, Y) = W_i(W_j(X, Y), Y), \qquad \textit{for every } i, j \geqq 0;$$

(b) *For every* $z \in \langle X, Y \rangle$ *and every* $i \geqq 0$, $W_i(X^z, Y^z) = W_i(X, Y)^z$.

(c) *Let* $T_0 = X^{-1}YX$, $T = T_0Y$, $X' = Y^{-T_0}$, $Y' = Y^T$, *and* $n \geqq 4$. *Then*

(i) $W_3(X, Y)^T = X'$,

(ii) $W_n(X, Y)^T = W_{n-3}(X', Y')$.

(d) *Let* $C_n(X, Y) = W_n(X, Y)Y$. *Then* $C_n(X, Y)^T = C_{n-3}(X', Y')$ *and* $[X, nY]^T = [X', (n-3)Y']$.

PROOF. Parts (a) and (b) are clear. We prove parts (c) and (d).

(c)(i) $\qquad W_3(X, Y) = Y^{-W_2(X, Y)}$ (by part (b))

$\qquad = Y^{(X^{-1}Y^{-1}X)}Y^{-1}Y^{-(X^{-1}Y^{-1}X)}$ (by part (a)).

Consequently,

$$W_3(X, Y)^T = Y^{(X^{-1}Y^{-1}X)T}Y^{-T}Y^{-(X^{-1}Y^{-1}X)T}$$
$$= Y^YY^{-T}Y^{-Y} \qquad \text{(by the definition of } T)$$
$$= Y \cdot Y^{-T}Y^{-1}$$
$$= Y^{-TY^{-1}}$$
$$= Y^{-T_0} \qquad \text{(by the definition of } T_0),$$

i.e., $W_3(X, Y)^T = X'$, as required.

(ii) $W_n(X, Y)^T = W_{n-3}(W_3(X, Y)^T, Y^T)$ (by parts (a) and (b))

$= W_{n-3}(X', Y')$ (by (i) and the definition of Y')

i.e., $W_n(X, Y)^T = W_{n-3}(X', Y')$, as required.

(d) $C_n(X, Y)^T = W_n(X, Y)^T Y^T = W_{n-3}(X, Y)Y'$, (by (c)(ii) and the definition of Y'). Consequently,

$$(*) \qquad C_n(X, Y)^T = C_{n-3}(X', Y').$$

Finally, observe that $[X, nY] = C_n(X, Y)^{Y^{n-1}}$. Hence

$$\begin{aligned}[X, nY]^T = C_n(X, Y)^{Y^{n-1}T} &= C_n(X, Y)^{T \cdot T^{-1}Y^{n-1}T} \\ &= C_n(X, Y)^{T \cdot (Y^T)^{n-1}} \\ &= C_{n-3}(X', Y')^{Y'^{n-1}} \qquad \text{(by (*))} \\ &= [X', (n-3)Y'],\end{aligned}$$

i.e., $[X, nY]^T = [X', (n-3)Y']$ as required.

1.1.2. For further applications it is convenient to introduce some variations of the function δ. For every natural number k, define $\varepsilon_k : \{1, \ldots, 2^{k+1} - 1\} \to \{1, -1\} \subseteq \mathbf{Z}$ by

$$(1) \qquad \varepsilon_k(i) = \begin{cases} -\delta(i) & i \text{ odd}, \\ \delta(i) & i \text{ even}, \quad i \neq 2^k, \\ -1 & i = 2^k. \end{cases}$$

Then we can rewrite $W_k(X, Y)$ in the following uniform manner:

$$(2) \qquad W_k(X, Y) = X^{\varepsilon_k(1)} Y^{\varepsilon_k(2)} X^{\varepsilon_k(3)} \cdots X^{\varepsilon_k(2^{k+1}-1)}.$$

Finally, we fix n, $n \in \mathbf{Z}$, $n \geqq 1$ and define

$$(1') \qquad \varepsilon(i) = \begin{cases} \varepsilon_n(i), & 1 \leqq i \leqq 2^{n+1} - 1, \\ 1, & i = 2^{n+1}, \\ \varepsilon_n(i - 2^{n+1}), & 2^{n+1} + 1 \leqq i \leqq 2^{n+2} - 1. \end{cases}$$

Then

$$(2') \qquad W_n(X, Y) = X^{\varepsilon(1)} Y^{\varepsilon(2)} X^{\varepsilon(3)} \cdots X^{\varepsilon(2^{n+1}-1)}.$$

Now, it follows by an easy induction argument that $W_k(x, y) = w_k$. Hence we have

Proposition. *Let* $\varepsilon_k(i)$ *be defined by* (1), *set* $\varepsilon(i) = \varepsilon_n(i)$ *for* $1 \leqq i \leqq 2^{n+1} - 1$ *and let* $W_k(X, Y)$ *be defined by* (2). *Then*

(a) $w_n = W_n(x, y)$;

(b) *for* $0 \leqq l \leqq k \leqq n$, $w_k = W_{k-l}(w_l, y)$.

1.2. We turn now to the general case: F is a free group of finite rank and a, b are elements of F. We can certainly assume that $\langle a, b \rangle$ is not cyclic.

If there is no cancellation between $a^{\pm 1}$ and $b^{\pm 1}$ then $W_k(a, b)$ is reduced as written. However, if such a cancellation occurs, then this is no longer true. Theorem 1.2.3. shows how to overcome this difficulty if n is large enough.

1.2.1. *Definitions, notation and preliminary results*

Let F be a free group.

(a) A pair $(a, b) \in F \times F$ is *completely adequate* if all the products ab, ab^{-1}, $a^{-1}b$ and $a^{-1}b^{-1}$ are freely reduced.

(b) A pair (a, b) is *adequate* if the following conditions are satisfied:

  (i) At most one of the words ab, ab^{-1}, $a^{-1}b$ and $a^{-1}b^{-1}$ is not freely reduced; if $a^e b^\varepsilon$ is not reduced, then $a^e \neq a_0 b^{-\varepsilon}$, $e, \varepsilon \in \{1, -1\}$.

  (ii) If (a, b) is not completely adequate, then a and b are cyclically reduced.

(c) For a cyclically reduced word $w \in F$ denote by $\underline{\underline{w}}^*$ the set of all the cyclic conjugates of w.

(d) The pair $(a', b') \in F \times F$ is a cyclic conjugate of $(a, b) \in F \times F$ if there is an element $u \in F$ such that $a = a_0 u$, $a' = u a_0$, $b = b_0 u$ and $b' = u b_0$.

(e) $\mathrm{Can}(a, b)$ denotes the longest tail of a in reduced form which cancels out in the product ab. If no cancellation occurs, then $\mathrm{Can}(a, b) = 1$.

(f) For a reduced word $1 \neq w \in F$ let $h(w)$ be the first letter of w and let $t(w)$ be the last letter of w. Denote by $\mathscr{H}(w)$ the set of all the initial subwords of w and by $\mathscr{T}(w)$ the set of terminal subwords of w. Note that if $u \in \mathscr{H}(w)$ and $v \in \mathscr{H}(u)$ then $v \in \mathscr{H}(w)$. Similarly, if $u \in \mathscr{T}(w)$ and $v \in \mathscr{T}(u)$ then $v \in \mathscr{T}(w)$.

Lemma. *Let* $a, b \in F$ *be reduced words. Let* $u \in \mathscr{T}(a)$ *and* $v \in \mathscr{H}(b)$. *Then*

(i) $\mathrm{Can}(u, v) = 1$ *implies* $\mathrm{Can}(a, b) = 1$;

(ii) $\mathrm{Can}(u, v) \in \mathscr{T}(\mathrm{Can}(a, b))$.

Hence, if $\mathrm{Can}(a, b) = 1$, *then* $\mathrm{Can}(u, v) = 1$.

PROOF. Immediate by the definition.

(g) For $a \in F$ let $\rho(a)$ be the reduced word corresponding to a. Thus $\rho(a)$ is the shortest word in F which represents a. For a reduced word w let $|w|$ be the number of letters of w.

LEMMA. *Let a, b, g be reduced words in F. Then*
(i) $\mathrm{Can}(a^g, b^g) = \rho(\mathrm{Can}(a, b)g)$.
(ii) *If a and b are cyclically reduced and g is a head of a and g^{-1} is a tail of b then* $\mathrm{Can}(a^g, b^g) = \mathrm{Can}(a, b)\cdot g$.

PROOF. Immediate by the definition.

(h) LEMMA. *Let $(a, b) \in F \times F$ be adequate and let $d = \mathrm{Can}(a, b)$. Assume $d \neq 1$. Then*
(i) *(a^{-1}, b) is adequate;*
(ii) *$(a^{d^{-1}}, b^{d^{-1}})$ is adequate.*

PROOF. (i) Immediate by the definition.
(ii) By assumption

(1) $a = a_0 d, \quad b = d^{-1} b_0, \quad a_0 \in \mathscr{H}(a), \quad b_0 \in \mathscr{T}(b)$ and $\mathrm{Can}(a_0, b_0) = 1.$

Hence, by Lemma 1.2.1(e)(ii), $\mathrm{Can}(b_0, a_0) \in \mathscr{T}(\mathrm{Can}(b, a))$. Since $\mathrm{Can}(b, a) = 1$, we get

$$\mathrm{Can}(b_0, a_0) = 1. \tag{2}$$

Now $a^{d^{-1}} = d(a_0 d)d^{-1} = da_0$ and $b^{d^{-1}} = b_0 d^{-1}$. Denote $a^* = a^{d^{-1}}$, $b^* = b^{d^{-1}}$. Then

$$a^* = da_0, \quad b^* = b_0 d^{-1}, \quad a_0 \in \mathscr{H}(a), \quad b_0 \in \mathscr{T}(b). \tag{3}$$

Since $\mathrm{Can}(a_0, b_0) = 1$, by Lemma 1.2.1(e)(i), $\mathrm{Can}(a^*, b^*) = 1$. $\mathrm{Can}(b^*, a^*) = \mathrm{Can}(b_0, a_0)d = d$, by (2). $\mathrm{Can}(a^{*-1}, b^*) = \mathrm{Can}(a_0^{-1} d^{-1}, b_0 d^{-1})$. Since b^* is cyclically reduced, $\mathrm{Can}(d^{-1}, b_0) = 1$, hence $\mathrm{Can}(a^{-1}, b^*) = 1$. Finally, $\mathrm{Can}(a^*, b^{*-1}) = \mathrm{Can}(da_0, db_0^{-1})$. Noting that a^* is cyclically reduced we see that $\mathrm{Can}(a_0, d) = 1$. Consequently $\mathrm{Can}(a^*, b^{*-1}) = 1$.

The Lemma is proved.

1.2.2. THEOREM. *Let $a, b \in F$ be reduced words, b cyclically reduced and assume that $\langle a, b\rangle$ is not cyclic. Fix $n \in \mathbf{N}$. Let*

$$\mathscr{A} = \{a^h \mid \langle [a^h, nb^*]\rangle^F = \langle [a, nb]\rangle^F \text{ for some } b^* \in \underline{\underline{b}}^*\}.$$

Let $a_0 \in \mathscr{A}$ such that $|a_0|$ is minimal possible. Assume that (a, b^*) is not completely adequate for every $b^* \in \underline{\underline{b}}^*$. Then

(a) *a_0 is cyclically reduced;*

(b) *a_0 has a cyclic conjugate $a_0^* = a_0^u$ such that (a_0^u, b^u) is adequate.*

PROOF. (a) Assume $a_0^e b^\varepsilon$ is not reduced, $e, \varepsilon \in \{1, -1\}$ and let $t = t(\mathrm{Can}(a_0^e, b^\varepsilon))$. Then $b = t^{-1} b_0$. If a_0 is not cyclically reduced then $a_0 = t^{-1} a_1 t$. Let $b_1 = b^{t^{-1}}$. Then $b_1 = b_0 t^{-1} \in \underline{\underline{b}}^*$ and $[a_1, nb_1] = [a_0^{t^{-1}}, nb_0^{t^{-1}}]$. Consequently $a_1 \in \mathscr{A}$. But $|a_1| < |a_0|$, violating the choice of a_0. Thus a_0 is cyclically reduced.

(b) We may assume without loss of generality that $a = a_0$. Assume $a^e b^\varepsilon$ is not reduced and

$$(*) \qquad |\mathrm{Can}(a^e, b^\varepsilon)| \leqq |\mathrm{Can}(a'^e, b'^\varepsilon)| \qquad \text{for every } (a', b') \in \underline{\underline{(a, b)}}.$$

CLAIM. $a^{e'} b^{\varepsilon'}$ is reduced for every $e', \varepsilon' \in \{1, -1\}$ provided $(e', \varepsilon') \neq (e, \varepsilon)$.

PROOF OF THE CLAIM. We have to check three cases:

Case 1. $e' = -e$ and $\varepsilon' = -\varepsilon$.

Let $g = \mathrm{Can}(b^\varepsilon, a^e)$. If $g \neq 1$ then g^{-1} is a head of a^e and g is a tail of b^ε. Let $a^* = \rho((a^e) g^{-1})$ and let $b^* = \rho((b^\varepsilon)^{g^{-1}})$. Then

$$(1) \qquad (a^*, b^*) \in \underline{\underline{(a, b)}}.$$

On the other hand, by Lemma 1.2.1 $\mathrm{Can}(a^*, b^*) = \mathrm{Can}(a^e, b^\varepsilon) \cdot g^{-1}$. Hence $|\mathrm{Can}(a^*, b^*)| = |\mathrm{Can}(a^e, b^\varepsilon)|\,|g^{-1}| > |\mathrm{Can}(a^e, b^\varepsilon)|$, i.e.,

$$(2) \qquad |\mathrm{Can}(a^*, b^*)| > |\mathrm{Can}(a^e, b^\varepsilon)|.$$

But (1) together with (2) violates (*). Consequently $\mathrm{Can}(b^\varepsilon, a^e) = g = 1$.

Case 2. $e' = e$ and $\varepsilon' = -\varepsilon$.

Let $g = \mathrm{Can}(a^e, b^{-\varepsilon})$ and let $d = \mathrm{Can}(a^e. b^\varepsilon)$. If $g \neq 1$ then by definition

$$(3) \qquad \text{(i)}\ \ g \in \mathscr{T}(b^\varepsilon) \quad \text{and} \quad \text{(ii)}\ \ g \in \mathscr{T}(a^e);$$

$$(4) \qquad \text{(i)}\ \ d \in \mathscr{T}(a^e) \quad \text{and} \quad \text{(ii)}\ \ d^{-1} \in \mathscr{H}(b^\varepsilon).$$

Let $\mathscr{T} = \mathscr{T}(g) \cap \mathscr{T}(d)$. Since $g \neq 1$, by 3(ii) and 4(i), $\mathscr{T} \neq \{1\}$. Let $t \in \mathscr{T}$. Then by (3)(i) $t \in \mathscr{T}(b^\varepsilon)$ and by (4)(ii) $t^{-1} \in \mathscr{H}(b^\varepsilon)$. But then b^ε is not cyclically reduced. Thus $\mathrm{Can}(a^e, b^{-\varepsilon}) = g = 1$, as required.

Case 3. $e' = -e$ and $\varepsilon' = \varepsilon$.

Let $g = \mathrm{Can}(a^{e'}, b^{\varepsilon'})$. If $g \neq 1$ then the arguments of Case 2 show that a^e is not cyclically reduced. Thus $g = 1$, as required.

1.2.3. DEFINITION. Let $(a, b) \in F \times F$. The *associated pair* $s(a, b)$ is defined by $s(a, b) = (\rho(a'), \rho(b'))$, where $a' = a^{-1}b^{-1}ab^{-1}a^{-1}ba = b^{-a^{-1}ba} = w_2(a, b^{-1})$ and $b' = b^{-1}a^{-1}b^{-1}aba^{-1}bab = b^{a^{-1}bab} = w_2(a, b^{-1})^{-b}$.

We shall need the following simple Lemma in the sequel.

1.2.4. LEMMA. *Let $a, b \in F$ be cyclically reduced such that $\langle a, b\rangle$ is not cyclic. Suppose* $\mathrm{Can}(a, b) \neq 1$ *and, for* $e \neq 1$ *or* $\varepsilon \neq 1$, $\mathrm{Can}(a^{\varepsilon}, b^{e}) = 1$. *Then* $\rho(b^{-1}aba^{-1}) = uxv$ *where* $u \in \mathscr{H}(b^{-1})$ *and* $v \in \mathscr{T}(a^{-1})$ *(x possibly empty).*

PROOF. If $\mathrm{Can}(a, b) \neq a$ then the assertion is immediate. So assume $\mathrm{Can}(a, b) = a$. Let $\rho(ab) = u$. Then $u \in \mathscr{T}(b)$, hence $\mathrm{Can}(u, a^{-1}) \in \mathscr{T}(\mathrm{Can}(b, a^{-1}))$ by Lemma 1.2.1(e)(i). But since $\mathrm{Can}(b, a^{-1}) = 1$ by assumption, $\mathrm{Can}(u, a^{-1}) = 1$ by Lemma 1.2.1(e)(ii). Thus $\rho(aba^{-1}) = ua^{-1}$ and $\rho(b^{-1}aba^{-1}) = \rho(\rho(b^{-1}u)a^{-1}) = \rho(b^{-1}ua^{-1})$. If $\mathrm{Can}(b^{-1}, u) \neq u^{-1}$ then again our assertion is clear. So assume $\mathrm{Can}(b^{-1}, u) = u^{-1}$ and let $\rho(b^{-1}u) = v$. Then $v \in \mathscr{H}(b^{-1})$, $|v| = |a|$ and $\rho(b^{-1}aba^{-1}) = \rho(va^{-1})$. But since $\langle a, b\rangle$ is not cyclic, $\rho(b^{-1}aba^{-1}) \neq 1$ and hence $v \neq a$. The result follows.

1.2.5. THEOREM. *Let $(a, b) \in F \times F$ be an adequate pair and assume that $\langle a, b\rangle$ is not cyclic. Let $(\alpha, \beta) = s(a, b)$, $\alpha = \rho(a')$ and $\beta = \rho(b')$. Then*

- (i) (α, β) *is completely adequate*; $2(|a| + |b|) \leqq |\alpha| + |\beta| \leqq 8(|a| + |b|)$.
- (ii) *Assume that (a, b) is completely adequate. Then α and β can be decomposed by $\alpha = h^{-1}t^{-1}h$, $\beta = u^{-1}h^{-1}thu$ such that $|u|, |t| < |h|$.*
- (iii) *Assume $n \geqq 4$ and let $\gamma = [\alpha, (n-3)\beta]$. Then γ is reduced and $\gamma = [a, nb]^{\tau}$, where $\tau = a^{-1}bab$.*

PROOF. (i) We may assume that (a, b) is not completely adequate. Thus exactly one of ab, $a^{-1}b$, $a^{-1}b^{-1}$ and ab^{-1} is not reduced. By Lemma 1.2.3(g) we may assume that one of the following cases occur. *Case* 1: $\mathrm{Can}(a, b) \neq 1$; *Case* 2: $\mathrm{Can}(a^{-1}, b) \neq 1$. In both cases we shall prove that $h(a') = h(a^{-1})$, $t(a') = h(a)$, $h(b') = b^{-1}$ and $t(b') = b$. Since $\mathrm{Can}(a, b^{-1}) = 1$ and hence $\mathrm{Can}(b, a^{-1}) = 1$, this proves (i).

Case 1. $a' = a^{-1}b^{-1}(ab^{-1}a^{-1}b)a = a^{-1}b^{-1}(v^{-1}u^{-1})a$, $v^{-1} \in \mathscr{H}(a)$ and $u^{-1} \in \mathscr{T}(b)$, i.e. (by Lemma 1.2.4)

(1) $a' = a^{-1}b^{-1}v^{-1}u^{-1}a, \quad v^{-1} \in \mathscr{H}(a), \quad u^{-1} \in \mathscr{T}(b).$

Hence, by Lemma 1.2.1(e)(i), $\mathrm{Can}(b^{-1}, v^{-1}) \in \mathscr{T}(\mathrm{Can}(b^{-1}, a))$ and $\mathrm{Can}(u^{-1}, a) \in \mathscr{T}(\mathrm{Can}(b, a))$.

(2) $\mathrm{Can}(b^{-1}, v^{-1}) \in \mathscr{T}(\mathrm{Can}(b^{-1}, a))$ and $\mathrm{Can}(u^{-1}, a) \in \mathscr{T}(\mathrm{Can}(b, a))$.

But since (a, b) is adequate

(3) $\mathrm{Can}(b^{-1}, a) = \mathrm{Can}(b, a) = 1.$

Combining (2) and (3) with (1), and noting that $\mathrm{Can}(a^{-1}, b^{-1}) = 1$, yields $\rho(a') = a^{-1}b^{-1}v^{-1}u^{-1}a$, and the result follows.

Consider now b'. Thus $b' = (b^{-1}a^{-1})b^{-1}(ab)a^{-1}b(ab)$. We may assume that $b = a^{-1}t$. Then

(1) $b' = t^{-1}(t^{-1}ata^{-1})a^{-1}tt$, $t \in \mathscr{T}(b)$.

Since b is cyclically reduced,

(2) $\mathrm{Can}(t, a^{-1}) = 1$.

Since b^{-1} is reduced,

(3) $\mathrm{Can}(t^{-1}, a) = 1$.

If $\mathrm{Can}(a, t) = 1$, then our assertion is clear. So assume

(4) $\mathrm{Can}(a, t) = d \neq 1$.

Then by (2) and (4),

(5) t is cyclically reduced.

Consequently, it follows from (2), (3), (4), (5) and Lemma 1.2.4 that

(6) $\rho(t^{-1}ata^{-1}) = uxv$, $u \in \mathscr{H}(t^{-1})$, $v \in \mathscr{T}(a^{-1})$.

Combining (1)–(6) and noting that a is cyclically reduced, we get $\rho(b') = t^{-1}uva^{-1}tt$. Since $t \in \mathscr{T}(b)$ and $t^{-1} \in \mathscr{H}(b^{-1})$, the result follows.

Case 2. $a' = a^{-1}b^{-1}ab^{-1}a^{-1}ba$. We may assume $b = at$. Then $a' = a^{-1}t^{-1}a^{-1}at^{-1}a^{-1}a^{-1}ata$, i.e.,

(1) $a' = a^{-1}t^{-1}t^{-1}a^{-1}ta$, $t \in \mathscr{T}(b)$.

Since b is cyclically reduced,

(2) $\mathrm{Can}(t, a) = \mathrm{Can}(a^{-1}, t^{-1}) = \mathrm{Can}(t^{-1}, a^{-1}) = 1$.

If $\mathrm{Can}(a^{-1}, t) = 1$, then $\rho(a') = a'\rho(t^{-2})a^{-1}ta$ and we are done. Thus assume that

(3) $\mathrm{Can}(a^{-1}, t) \neq 1$.

Then it follows from (2) and (3) that

(4) t is cyclically reduced.

It follows from (2), (3), (4) and Lemma 1.2.4 that

(5) $\rho(t^{-1}a^{-1}ta)=uv$, $u\in\mathscr{H}(t^{-1})$, $v\in\mathscr{T}(a)$.

Consequently, by (1), (2), (4), and (5), $\rho(a')=a^{-1}t^{-1}u$. Here v and $u\in\mathscr{T}(a)$. This proves our assertion.

Finally, consider b'. We may assume $b^{-1}=ha^{-1}$. Then $b'=ha^{-1}a^{-1}hah^{-1}h^{-1}aah^{-1}$. Since b is cyclically reduced,

$$\mathrm{Can}(h,a^{-1})=\mathrm{Can}(a,h^{-1})=\mathrm{Can}(a^{-1},h)=\mathrm{Can}(h^{-1},a)=1,$$

hence if $\mathrm{Can}(h,a)=1$, then we are done. So assume $\mathrm{Can}(h,a)\neq 1$. Then since $\mathrm{Can}(h^{-1},a)=1$, we see that h is cyclically reduced. But then, by Lemma 1.2.4, $a^{-1}hah^{-1}=uv$, where $u\in\mathscr{H}(a^{-1})$ and $v\in\mathscr{T}(h^{-1})$. But then $\rho(b')=ha^{-1}uvh^{-1}aah^{-1}$, hence our result follows.

(ii) Take $t=b$, $h=a^{-1}ba$ and $u=b$.

(iii) γ is reduced by part (i) of the Theorem. Finally $[a,nb]^{\tau}=[\alpha,(n-3)\beta]$ by Proposition 1.1.1(c).

1.3. In this subsection we prove the necessary results on the function ε defined in 1.1. We begin with the following easy formulas.

1.3.1. Lemma. *Let $a=2^ka_0$, $b=2^lb_0$, a_0, b_0 odd, $k\leqq l$. Then the following hold*:

(a) $\varepsilon(a)=\varepsilon(a_0)$ for $a_0\leqq a\leqq 2^{n+2}-1$.

(b) $$\varepsilon(a\pm b)=\begin{cases}\mp\varepsilon(a), & l-k=1\\ \pm\varepsilon(a_0\pm b_0), & l-k=0\\ \pm\varepsilon(a), & l-k\geqq 2\end{cases}\quad \text{for } 1\leqq a+b\leqq 2^{n+2}-1.$$

(c) $$\varepsilon(2^{n+1}\mp a)=\begin{cases}\mp\varepsilon(a) & \text{if } a\neq 2^n\\ \pm\varepsilon(a) & \text{if } a=2^n\end{cases}\quad\text{for } 1\leqq a\leqq 2^{n+1}-1.$$

The proof is immediate. We therefore omit it.

1.3.2. Before formulating the main lemma we fix some notation.

(i) We shall denote by $[i,j]$ the interval $\{i,i+1,i+2,\dots,j\}$ for integers $i\leqq j$.

(ii) Let $[i_1,i_2],[j_1,j_2]\subseteq[1,2^{n+2}]$ and assume that $|[i_1,i_2]|=|[j_1,j_2]|$, i.e., $j_1-i_1=j_2-i_2$. We shall denote the fact that $\varepsilon(i_1+t)=\varepsilon(j_1+t)$ for t, $0\leqq t\leqq i_2-i_1$ by $\varepsilon(i_1,i_2)=\varepsilon(j_1,j_2)$. Similarly, if $\varepsilon(i_1+t)=-\varepsilon(j_2-t)$ for t, $0\leqq t\leqq i_2-i_1$, we shall write $\varepsilon(i_1,i_2)=-\varepsilon(j_1,j_2)$. If $I=[i_1,i_2]$ we shall write $\varepsilon(I)$ for $\varepsilon(i_1,i_2)$.

(iii) Let a be an integer. If $a=2^\alpha a_0$, a_0 being odd, we shall define $\nu(a)=\alpha$.

1.3.3. LEMMA. *Let* $[i_1, i_2], [j_1, j_2] \subseteq [1, 2^{n+2}]$, $i_1 < j_1$, $i_1 \equiv j_1 \bmod 2$ *and assume that* $\varepsilon(i_1, i_2) = \varepsilon(j_1, j_2)$, $(\varepsilon(i_1, i_2) = -\varepsilon(j_1, j_2))$. *Consider* $v(t)$ *and* $v(s)$ *for* $t \in [i_1, i_2]$ *and* $s \in [j_1, j_2]$ *and let* $a \in [i_1, i_2]$, $b \in [j_1, j_2]$ *such that* $v(a) = \max_{t \in [i_1, i_2]} v(t)$ *and* $v(b) = \max_{s \in [j_1, j_2]} v(s)$. *These conditions define a and b uniquely. Let* $\alpha = v(a)$ *and* $\beta = v(b)$. *Then*

(i) $a - i_1 = b - j_1$ $(a - i_1 = j_2 - b)$.

(ii) *If* $v(t) < \alpha - 1, \beta - 1$, *then* $\varepsilon(a - t) = \varepsilon(b - t)$, $\varepsilon(a + t) = \varepsilon(b + t)$, $\varepsilon(a - t) = -\varepsilon(b + t)$.

(iii) *Let* $\mathscr{P}([i_1, i_2], [j_1, j_2])$ *be the set of all the pairs of intervals* (I, J), $I, J \subseteq [1, 2^{n+2}]$ *such that* $[i_1, i_2] \subseteq I$, $[j_1, j_2] \subseteq J$ *and* $\varepsilon(I) = \varepsilon(J)$ $(\varepsilon(I) = -\varepsilon(J))$ *and let* $[I', J']$ *be the maximal element of* $\mathscr{P}([i_1, i_2], [j_1, j_2])$ *with respect to inclusion. Then there exist* $a' \in I'$, $b' \in J'$ *and a number* γ, $1 \leqq \gamma \leqq n$ *such that*

$$I' = [a' - (2\gamma - 1), a' + (2\gamma - 1)], \qquad J' = [b' - (2\gamma - 1), b' + (2\gamma - 1)]$$

(*i.e.,* a' *and* b' *are in the middle of* I' *and* J' *respectively*).

(1) *If* $\alpha < \beta$ *then* $\gamma = \alpha - 1$, $a' = a$, $b' = b$.

(2) *If* $\alpha > \beta$ *then* $\gamma = \beta - 1$, $a' = a$, $b' = b$.

(3) *If* $\alpha = \beta$ *then* $\gamma \geqq \alpha$ *and* $[a - (2^{\alpha-1} - 1), a + 2^{\alpha-1} - 1] \subseteq I$.

(*) The following Corollary determines the intervals that correspond to occurrences of W_i in R.

COROLLARY. *Let* $I \subseteq [1, 2^{n+2}]$. *Then*

(1) $\varepsilon(I) = \varepsilon(1, 2^e - 1)$ *if and only if*

$$I = [d - (2^{e-1} - 1), d + (2^{e-1} - 1)]$$

for some $d \in I$ *satisfying* $d = 2^{e-1} f$, $f \equiv 1 \bmod 4$.

(2) $\varepsilon(I) = -\varepsilon(1, 2^{e-1})$ *if and only if*

$$I = [d - (2^{e-1} - 1), d + (2^{e-1} - 1)]$$

for some $d \in I$ *satisfying* $d = 2^{e-1} g$, $g \equiv 3 \bmod 4$.

PROOF. (i) Although the proofs for $\varepsilon(i_1, i_2) = \varepsilon(j_1, j_2)$ and $\varepsilon(i_1, i_2) = -\varepsilon(j_1, j_2)$ are essentially the same, we prove them separately, for convenience. Assume $\alpha \leqq \beta$ and $\varepsilon(i_1, i_2) = \varepsilon(j_1, j_2)$. Let $c = i_1 + b - j_1$. If our claim does not hold, then $c \neq a$, hence $\omega := v(c) < \alpha \leqq \beta$ and $\beta - \omega \geqq 1$. Since $i_1 \equiv j_1 \bmod 2$, $\omega = 0$ if and only if b is odd, i.e., $\beta = v(b) = 0$, contradicting $\beta \geqq 1$. So we have

$$(*) \qquad 0 < \omega < \alpha \leqq \beta.$$

Since $\nu(c \pm 2^{\omega-r}) = \omega - r$ for $r > 0$ and by (*) $\omega < \alpha$, we must have either $c + 2^{\omega-1} \in [i_1, i_2]$ and then $b + 2^{\omega-1} \in [j_1, j_2]$ or $c - 2^{\omega-1} \in [i_1, i_2]$ and then $b - 2^{\omega-1} \in [j_1, j_2]$. (Note that $\omega - 1 \geqq 0$ by (*) hence $2^{\omega-1}$ is an integer.) But $c \pm 2^{\omega-1} = 2^{\omega-1}(2k \pm 1)$, k being odd, while $b \pm 2^{\omega-1} = 2^{\omega-1}(h_0 2^{\beta-\omega+1} \pm 1) = 2^{\omega-1}(4h \pm 1)$, h being odd, $h \in \mathbf{Z}$. Hence

$$(**) \qquad \varepsilon(c \pm 2^{\omega-1}) = \mp 1 \qquad \text{while } \varepsilon(b \pm 2^{\omega-1}) = \pm 1.$$

However, by the assumption, one of the equations $\varepsilon(c \pm 2^{\omega-1}) = \varepsilon(b \pm 2^{\omega-1})$ must hold, violating (*). Consequently $c = a$, i.e., $a - i_1 = b - j_1$.

Now assume $a \leqq \beta$ and $\varepsilon(i_1, i_2) = -(j_1, j_2)$. Let $c = i - b + j$. If $c \neq a$ then (*) holds. Hence, as in the first case, we must have either $c + 2^{\omega-1} \in [i_1, i_2]$ and then $b - 2^{\omega-1} \in [j_1, j_2]$ or $c - 2^{\omega-1} \in [i_1, i_2]$ and then $b + 2^{\omega-1} \in [j_1, j_2]$. As above, this leads to (**). However, by assumption one of the equations $\varepsilon(b \pm 2^{\omega-1}) = -\varepsilon(c \mp 2^{\omega-1})$ must hold, i.e., $\varepsilon(b \pm 2^{\omega-1}) = \varepsilon(c \pm 2^{\omega-1})$, by Lemma 1.3.1(b). This again violates (**). Hence $c = a$. If $\alpha \geqq \beta$, the same arguments give the results simply by replacing α by β and β by α.

(ii) Immediate by Lemma 1.3.1(b).

(iii) Assume first the $\varepsilon(I) = \varepsilon(J)$ and $\alpha \leqq \beta$. For $a, t \in \mathbf{N}$ let

$$I_t(a) = [a + t \cdot 2^{\alpha-1} + 1, a + (t+1)2^{\alpha-1} - 1]$$

and

$$I_{-t}(a) = [a - (t+1) \cdot 2^{\alpha-1} + 1, a - t \cdot 2^{\alpha-1} - 1]$$

whenever they are contained in $[1, 2^{n+2}]$ and undefined otherwise. By part (ii)

$$(*) \qquad \varepsilon(I_t(a)) = \varepsilon(I_t(b)) \quad \text{and} \quad \varepsilon(I_{-t}(a)) = \varepsilon(I_{-t}(b)),$$

in particular $\varepsilon(I_{-0}(a) \cup \{a\} \cup I_0(a)) = \varepsilon(I_{-0}(b) \cup \{b\} \cup I_0(b))$, i.e.,

$$(**) \quad \varepsilon(a - 2^{\alpha-1} + 1, a + 2^{\alpha-1} - 1) = \varepsilon(b - 2^{\alpha-1} + 1, b + 2^{\alpha-1} - 1).$$

Assume that $\alpha < \beta$. Then $\varepsilon(a \pm 2^{\alpha-1}) = \mp 1$ while $\varepsilon(b + 2^{\alpha-1}) = \pm 1$ by Lemma 1.3.1(b). Hence in view of (**), parts (iii)(1) and (iii)(2) follow. Assume $\alpha = \beta$. Then $\varepsilon(a \pm 2^{\alpha-1}) = \varepsilon(b \pm 2^{\alpha-1}) = \mp 1$, hence by (*)

$$(***) \qquad \varepsilon(a - 2^{\alpha} + 1, a + 2^{\alpha} - 1) = \varepsilon(b - 2^{\alpha} + 1, b + 2^{\alpha} - 1).$$

If $I' = [a - 2^{\alpha} + 1, a + 2^{\alpha} - 1]$ we are done. So assume to the contrary and let $a' \in I'$, $b' \in J'$ with $\nu(a') = \alpha'$, $\nu(b') = \beta'$, and let $a' = \max_{t \in I'} \nu(t)$, $\beta' = \max_{s \in J'} \nu(s)$. Clearly $\alpha' \geqq \alpha$ and $\beta' \geqq \beta$. If $\alpha' < \beta'$ then

$$I' = [a' - 2^{\alpha'-1} + 1, a' + 2^{\alpha'-1} - 1] \quad \text{and} \quad J' = [b - 2^{\alpha'-1} + 1, b + 2^{\alpha'-1} -],$$

by parts (1) and (2) of (iii). By assumption $\alpha' > \alpha$ as I' strictly contains $[a' - 2^{\alpha} + 1, a' + 2^{\alpha} - 1]$. Hence $a' - 1 \geqq \alpha$ and part (3) of (iii) holds for this case. Similar argument leads to the result of $\alpha' > \beta'$. Assume $\alpha' = \beta'$. Then by (***) $[a' - 2^{\alpha'} + 1, a' + 2^{\alpha'} - 1] \subseteq I'$. By the maximality of I' and the maximality of α' this implies $I' = [a' - 2^{\alpha'} + 1, a' + 2^{\alpha'} - 1]$, for otherwise $a' - 2^{\alpha'}$ or $a' + 2^{\alpha'}$ should be in I'. This completes the proof for the case $\varepsilon(I) = \varepsilon(J)$. The case $\varepsilon(I) = -\varepsilon(J)$ is treated similarly.

Proof of the Corollary. Let $a \in [1, 2^e - 1]$, $b \in I$ such that $\nu(a)$ is maximal in $\{\nu(t), 1 \leqq t \leqq 2^e - 1\}$ and $\nu(b)$ is maximal in $\{\nu(s) \mid s \in I\}$. Then $a = 2^{\varepsilon - 1}$ and, by part (iii)(1), $b = k \cdot 2^{e-1}$, k being odd. Since $\varepsilon(b) = \varepsilon(a) = 1$, we must have $\varepsilon(k) = 1$ by Lemma 1.3.1(a). Consequently, $k \equiv 1 \bmod 4$. Conversely, if $k \equiv 3 \bmod 4$ then $\varepsilon(b) = -1$ while $\varepsilon(a) = 1$, hence $\varepsilon(I) \neq \varepsilon(1.\, 2^e - 1)$. The rest is treated similarly.

1.4. In this subsection we describe the maximal pieces (Theorem 1.4.2) for the special case $F = \langle x, y \rangle$ and $R = [x, ny]$.

1.4.1. *Definitions and notations*

(a) Let $F = \langle x, y \rangle$. Denote by $\mathscr{R}$ the set of all the (reduced) cyclic conjugates of $w_n y$ and $y^{-1} w_n^{-1}$, $w_n = W_n(x, y)$.

(b) A word c in F is a *piece* (with respect to $\mathscr{R}$) if there are elements r_1 and r_2 in $\mathscr{R}$, $r_1 \neq r_2^{-1}$ such that $r_1 = ac$, $r_2 = c^{-1}b$. It is a *maximal piece* if ab and ba are both reduced. Thus c is a maximal piece if after cancelling out c in the product $r_1 r_2$, we get a word which is cyclically reduced.

(c) For $1 \leqq i \leqq n$ let

$$P_i = \{w_{i-1}^{-1} y w_{i-1}, w_{i-1}^{-1} y^{-1} w_{i-1}, w_{i-1} y w_{i-1}^{-1}, w_{i-1} y^{-1} w_{i-1}^{-1}\}.$$

1.4.2. Theorem. *Let $c \in F$. c is a maximal piece with respect to $\mathscr{R}$ if and only if $c \in P_i$ for some i.*

Proof. Let $r_1, r_2 \in \mathscr{R}$. It is enough to consider two cases.

Case I. r_1 and r_2 are cyclic conjugates of $w_n y$.
Let

$$w_n = \prod_{i=1}^{2^{n+1}-1} a_i^{\varepsilon(i)},$$

$a_i = x$ for i odd and $a_i = y$ for i even (see 1.1). Then

$$c_n^2 = w_n y w_n y = \prod_{i=1}^{2^{n+2}} a_i^{\varepsilon(i)},$$

a_i being as above (see 1.1(1′)). For $1 \leqq i, j \leqq 2^{n+2}$ we shall denote by $R^+[i,j]$ the subword $a_i^{\varepsilon(i)} \cdots a_j^{\varepsilon(j)}$ of c_n^2.

Let $r_1 = ac$, $r_2 = c^{-1}b$, $r_1 r_2^{-1} \neq 1$. Let $c = R^+[i_1, i_2]$ as a subword of r_1 and let $c^{-1} = R^+[j_1, j_2]$ as a subword of r_2. Then in the notation of 1.3 we must have $\varepsilon(j_1, j_2) = -\varepsilon(i_1, i_2)$. Assume c is a maximal piece, and let $a = R^+[i, i_1 - 1]$, $b = R^+[j_2 + 1, j]$. Then $\varepsilon(i_1 - 1) \neq -\varepsilon(j_2 + 1)$ and $\varepsilon(i) \neq -\varepsilon(j)$ (see Fig. 1).

Fig. 1.

Since $i_2 = i - 1 + 2^{n+1}$, $j = j_1 - 1 + 2^{n+1}$, $\varepsilon(i) = \varepsilon(i_2 + 1)$ and $\varepsilon(j) = \varepsilon(j_1 - 1)$ thus $\varepsilon(i_1 - 1) \neq -\varepsilon(j_2 + 1)$, $\varepsilon(i_2 + 1) \neq -\varepsilon(j_1 - 1)$. Consequently $([i_1, i_2], [j_1, j_2])$ is the maximal element of $P([i_1, i_2], [j_1, j_2])$ (which actually consists only of that element) in the notation of 1.3. By Lemma 1.3.3(iii) this implies that $[i_1, i_2] = [k - (2^\alpha - 1), k + (2^\alpha - 1)]$ for some $k = 2^{\alpha+1} \cdot q$, $q \in \mathbf{Z}$ and $[j_1, j_2] = [l - (2^\alpha - 1), l + 2^\alpha - 1]$, $l = 2^{\alpha+1}s$, $s \in \mathbf{Z}$. Therefore $c = w_{\alpha-1}^{-1} y^{\pm 1} w_{\alpha-1}$ by Lemma 1.3.3(i), as required. On the other hand, it follows easily from Lemma 1.3.3(iii), (iv) how to define elements $r_1, r_2 \in \mathcal{R}$ for which the elements of the P_i are maximal pieces.

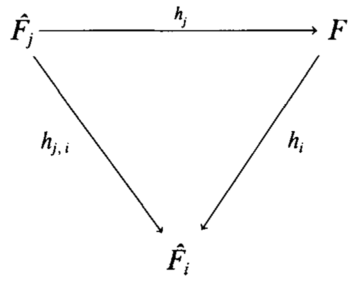
Case II. r_1 and r_2 are cyclic conjugates of $w_n c$ and $c^{-1} w_n^{-1}$ respectively.

We shall number the letters of c_n as in Case I. Hence if $c_n = a_1^{\varepsilon(1)} \cdots a_{2^{n+1}}^{\varepsilon(2^{n+1})}$ then $c_n^{-1} = a_{2^{n+1}}^{-\varepsilon(2^{n+1})} \cdots a_1^{-\varepsilon(1)}$. For $1 \leqq j_1 \leqq j_2 \leqq 2^{n+2}$ denote by $R^-[j_1, j_2]$ the subword $a_{j_2}^{-\varepsilon(j_2)} \cdots a_{j_1}^{-\varepsilon(j_1)}$ of c_n^{-2}. Let $r_1 = ac$, $r_2 = c^{-1}b$ and let $c = R^+[i_1, i_2]$ as a subword of r_1 and let $c^{-1} = R^-[j_1, j_2]$ as a subword of r_2. Then we must have $\varepsilon(i_2) = -(-\varepsilon(j_2))$,

$$\begin{aligned} \varepsilon(i_2 - 1) &= -(-\varepsilon(j_2 - 1)) \cdots \varepsilon(i_2 - t) \\ &= -(-\varepsilon(j_2 - t)) \cdots \varepsilon(i_1) = -(-\varepsilon(j_1)), \end{aligned}$$

$0 \leqq t \leqq i_2 - i_1$. Hence in the notation of 1.3 we must have $\varepsilon(i_1, i_2) = \varepsilon(j_1, j_2)$. Now the proof of Case II can be completed in a way similar to that of Case I using the equation $\varepsilon(i_1, i_2) = \varepsilon(j_1, j_2)$ instead of $\varepsilon(i_1, i_2) = -\varepsilon(j_1, j_2)$ in Lemma 1.3.3(iii).

1.4.3. For $0 \leqq j \leqq n$ let $\hat{F}_j$ be the free group on the two generators ω_j and η_j. Then we have monomorphisms $h_j : \hat{F}_j \to F$ and for $i \leqq j$ monomorphisms $h_{j,i} : \hat{F}_j \to \hat{F}_i$ defined by $h_{j,i}(\omega_j) = W_{j-i}(\omega_i, \eta_i)$, $h_{j,i}(\eta_j) = \eta_i$, $h_i(\omega_i) = w_i$ and $h_i(\eta_i) = y$. We have the following commutative diagram:



Theorem. *Let $a \in \hat{F}_j$ and let $h_j(a) = v_1 w_i^{\pm 1} v_2$, $v_1, v_2 \in F$, be a decomposition of $h_j(a)$ in F, $i \leqq j$. Then there is a unique decomposition of $h_{j,i}(a)$ by $h_{j,i}(a) = t_1 \omega_i^{\pm 1} t_2$ such that $h_i(t_1) = v_1$ and $h_i(t_2) = v_2$. In other words, every decomposition of $h_j(a)$ of the form $h_j(a) = v_1 w_i^{\pm 1} v_2$ can be lifted to $\hat{F}_i$.*

Proof. We may assume that $h_j(a) = v_1 w_i v_2$ and that $i > 0$. Since h_i is a monomorphism and $\hat{F}_i$ is a free group, if $h_{j,i}(a)$ has two decompositions $t_1 w_i t_2$ and $t_1' w_i t_2'$ such that $h_i(t_1) = h_i(t_1') = v_1$ and $h_i(t_2) = h_i(t_2') = v_2$ then they must coincide. Hence we have to show the existence of such a decomposition. If there is a $t_1 \in \hat{F}_i$ with $h_i(t_1) = v_1$ then t_2 with $t_2 = w_i^{\mp 1} t_1^{-1} h_{j,i}(a)$ satisfies $h_i(t_2) = v_2$. Thus if the theorem is false, we may asume $v_1, v_2 \notin h_i(\hat{F}_i)$. Since $h_{j,i}(a) \in h_i(\hat{F}_i)$, there are $s_1, s_2 \in \hat{F}_i$ such that $v_1 = h_i(s_1)u_1$, $v_2 = u_2 h_i(s_2)$, $u_1, u_2 \in F \setminus h_i(\hat{F}_i)$. Then $u_1 w_i u_2 \in h_i(\hat{F}_i)$. In particular $u_1 w_i = w_i^{\varepsilon} p_1$ for some $p_1 \in F$, $\varepsilon = \pm 1$. Assume $\varepsilon = 1$. Then $u_1 w_i = w_i p_1$, hence $w_i = u_1 u_3$ and $w_i = u_3 p$. Assume $|u_3| \leqq |w_{i-1}|$. Then $w_{i-1}^{-1} = u_3 u_4$, $u_4 \in F$, hence $w_{i-1} = u_4^{-1} u_3^{-1}$ and $w_{i-1} = u_5 u_3$, $u_5 \in F$. Since $|u_3| = |u^{-1}|$ we have $u_3^{-1} = u_3$, a contradiction, as F_3 is a free group.

Definition. Let W be a reduced word and let U be a subword of W. An *occurrence of U in W* is a triple $\langle h_W(U), U, t_W(U) \rangle$ of subwords $h_W(U)$, U and $t_w(U)$ of W such that the following hold:

(1) $W = h_W(U) U t_W(U)$ and

(2) $\mathrm{Can}(h_W(U), U) = \mathrm{Can}(U, t_W(U)) = 1$.

Denote by $\mathrm{Occ}(W)$ the set of all the occurrences of subwords of W in W. If we are not interested in a specified occurrence of a subword U of W in W, we shall write $U \in \mathrm{Occ}(W)$ instead of $\langle h_W(U), U, t_W(U) \rangle \in \mathrm{Occ}(W)$.

1.5.1. Definition. Let W be a reduced word in F. Let U, V and D be $\in \mathrm{Occ}(W)$.

D is the *overlap* between U and V if the following hold:

(i) $D \in \mathscr{T}(U) \cap \mathscr{H}(V)$;

(ii) If $I \in \mathscr{H}(U)$ and $T \in \mathscr{T}(V)$ such that $U = ID$ and $V = DT$, then IDT is a subword of W.

We shall denote $D = \mathrm{Ov}_G(U, V)$. Denote $I = h(U, V)$ and $T = t(U, V)$. If the group in which we consider the overlap is clear from the context, we shall simply write $\mathrm{Ov}(U, V)$.

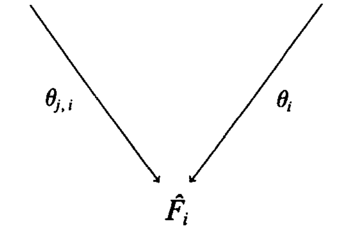
Lemma. (1) *Let U and U^{-1} be subwords of the reduced word W. Then $\mathscr{T}(U) \cap \mathscr{H}(U^{-1}) = 1$.*

(2) *Let A and A_2 be two occurrences of $A \in F$ in the reduced word $W \in F$. If $\mathrm{Ov}(A_1, A_2) \neq 1$ then $h(A_1, A_2) = ST$, $\mathrm{Ov}(A_1, A_2) = (ST)^\alpha S$ for some $\alpha \geqq 0$ and $t(A_1, A_2) = TS$.*

(3) *If $A^2 = A_0AA_3$ then $A_0, A_3, A \in \langle D \rangle$ for some $D \in F$.*

Proof. All parts are immediate and well known. We omit proofs.

1.5.2. Let F_0 be the free group generated by X and Y. Define $U(X, Y) = Y^{-X^{-1}YXY^{-1}X^{-1}Y^{-1}X}$. Let F be a finitely generated free group, n a natural number, $n \geqq 7$. Let x, y be reduced words in F such that $\langle x, y \rangle$ is not cyclic. By Theorem 1.2.2 there is a completely adequate pair $(\alpha, \beta) \in F \times F$ such that $\langle [x, ny] \rangle^F = \langle [\alpha, (n-3)\beta] \rangle^F$. Let $u = U(\alpha, \beta)$. Since $(u^{\alpha^{-1}\beta\alpha\beta}, \beta^{-\alpha^{-1}\beta\alpha}) = s(\alpha, \beta)$, clearly $\langle [u, (n-6)\beta] \rangle^F = \langle [\alpha, (n-3)\beta] \rangle^F$. Let $w_i = W_i(u, \beta)$ and define subgroups F_i of F by $F_i = \langle w_i, \beta \rangle$, $0 \leqq i \leqq n$. For $0 \leqq j \leqq n$ let $\hat{F}_j$ be the free group generated by ω_j and η_j. Then we have monomorphisms $\theta_j : \hat{F}_j \to F$ and for $i \leqq j$ monomorphisms $\theta_{j,i} : \hat{F}_j \to \hat{F}_i$ defined by $\theta_{j,i}(\omega_j) = W_{j-i}(\omega_i, \eta_i)$, $\theta_{j,i}(\eta_j) = \eta_i$ and $\theta_{j,i}(\eta_j) = \eta_i$, $\theta_i(\omega_i) = w_i$ and $\theta_i(\eta_i) = \beta$. We have the following commutative diagram:



THEOREM. *Let $w \in \hat{F}_j$ and let $\theta_j(w) = v_1 w_i^\varepsilon v_2$, $v_1, v_2 \in F$, $\varepsilon \in \{1, -1\}$ be a decomposition of $\theta_j(w)$ in F, $1 \leqq i \leqq j$. Then there is a unique decomposition of $\theta_{j,i}(w)$ by $\theta_{j,i}(w) = hw_i t$ such that $\theta_i(h) = v_1$ and $\theta_i(t) = v_2$. In other words, every decomposition of $\theta_j(w)$ of the form $\theta_j(w) = v_1 w_i v_2$ can be lifted to $\hat{F}_i$.*

PROOF. Without loss of generality we may assume that one of the following cases occurs.

Case 1. $w = \eta_j^k$, $k \in \mathbf{Z}$.

Case 2. $w = \eta_j^k \omega_i^e$, $k \in \mathbf{Z}$, $e \in \{1, -1\}$.

Case 3. $w = \omega_i^e \eta_j^k \omega_i^\delta$, $k \in \mathbf{Z}$, $e, \delta \in \{1, -1\}$.

Moreover, due to Proposition 1.1.1 we may assume that $j = i = 1$ mod $\varepsilon = 1$. By definition,

$$w_1 = u = \alpha^{-1}\beta\alpha\beta\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha = z^{-1}\beta^{-1}z,$$

where $z = \alpha^{-1}\beta\alpha\beta^{-1}\alpha^{-1}\beta^{-1}\alpha$.

Case 1. $\theta_1(w) = \beta^k$. Then both β and β^{-1} are subwords of β^k. This, however, contradicts Lemma 1.5.1(1).

Case 2. $\theta_1(w) = \beta^k w_1^e$. If the theorem does not hold, then $\beta^k w_1^e = \gamma w_1^e \mu$. Denote by p the occurrence of w_1^e in the left hand side and denote by q the occurrence of w_1^e in the right hand side. Let $I = h(p, q)$. Then $I = \mathrm{Ov}(\beta^k, q)$. Since I is a subword of β^k, $|I| \leqq |\alpha^{-1}\beta\alpha|$, otherwise a successive application of Lemma 1.5.1(3) would imply that $\alpha, \beta \in \langle c \rangle$ for some $c \in F$. This would, however, violate the assumption that $\langle x, y \rangle$ is not cyclic. Thus $|I| < \frac{1}{2}|p|$, hence $\mathrm{Ov}(p, q) > \frac{3}{4}|p|$. Consequently, it follows from Lemma 1.5.1(2) that $p = (ST)^s S$, $s \geqq 4$. On the other hand, $p = z^{-1}\beta^{-1}z$, $|z| > 2|\alpha^{-1}\beta\alpha|$. But then $z = (ST)^2 S_0$ and $z^{-1} = S_1(TS)^2$, $S_1 \in \mathscr{T}(S)$, hence we have simultaneously $ST \in \mathscr{H}(z)$ and $S^{-1}T^{-1} \in \mathscr{H}(z)$, contradicting Lemma 1.5.1(2).

Case 3. $\theta_1(w) = w_1^e y^k w_1^\delta$. If the theorem does not hold, then $w_1^e y^k w_1^\delta = \gamma w_1 \mu$. Let p_1 and p_2 be the occurrences of w_1 on the left hand side and let q be the occurrence of w_1 on the right hand side. Then we may assume $\mathrm{Ov}(p_1, q) \neq 1$ and $\mathrm{Ov}(q, p_2) \neq 1$, otherwise our case reduces to one of the previous cases. But since $w_1 = z^{-1}\beta^{-1}z$, it follows from Lemma 1.5.1(1) that $|h(p_1, q)| \leqq |\beta|$ and similarly $|t(q, p_2)| \leqq |\beta|$. However, then q is a subword of $p_1\beta^{\delta_1}$, $\delta_1 \in \{1, -1\}$, coming back to Case 2. This completes the proof of the theorem.

1.5.3. Successive application of Theorem 1.5.2 yields the following

THEOREM. *Let V be a reduced word in F such that $PVQ \in F_i$ for some P, Q in F, and assume that V contains w_i^ε, $i \geqq 1$, as a subword for some $\varepsilon \in \{1, -1\}$.*

Then V contains a unique maximal subword V_0 such that $V_0 \in F_i$, i.e., every subword V_0 of V which is in F_i is already a subword of V_0.

1.5.4. The following is the main result of Section 1.

THEOREM. *Let the notation be as in 1.5.2. Let $\beta = s^{-1}bs$, b cyclically reduced. Define $z_i = sw_is^{-1}$. Then z_i is reduced. Define sets P_i by*

$$P_i = \{z_{i-1}^{-1}bz_{i-1},\ z_{i-1}^{-1}b^{-1}z_{i-1},\ z_{i-1}bz_{i-1}^{-1},\ z_{i-1}b^{-1}z_{i-1}^{-1}\}.$$

Let $\mathscr{R}$ be the symmetric closure of $R = W_n(z_1, \beta)$ in F and let $w \in F$, $|w| \geqq |w_7|$. Then w is a maximal piece with respect to $\mathscr{R}$ if and only if $w \in P_i$ for some i.

PROOF. Immediate by Theorems 1.4.2 and 1.5.2.

2. The derived diagrams

2.1.1. *Notation and definitions*

Let M be an $\mathscr{R}$-diagram (see [8, V]) and let $v_1\mu v_2$ be a path in M where v_1 and v_2 are the initial and terminal vertices of $\bar{\mu}$ respectively. Denote $o(\mu) = v_1$ and $t(\mu) = v_2$. Here $\bar{\mu}$ stands for the closure of μ in M.

For a path μ in M, denote by Head(μ) the set of all the initial subpaths of μ and byTail(μ) the set of all the terminal subpaths of μ. Let Path(M) be the set of all the paths of M.

If M is an annular map, denote by $\omega(M)$ the outer boundary and by $\tau(M)$ the inner boundary of M.

Let M be an $\mathscr{R}$-diagram over F with labeling function $\phi = \phi_F$. Denote by Reg(M) the set of all the regions of M and let $\widehat{\text{Reg}}(M) = \text{Reg}(M) \cup \{D_\infty\}$, where D_∞ is the complement of M in E^2. Let Boreg(M) be the set of the boundary regions of M. For submaps K and L of M let Ov(K, L) be the set of non-trivial connected components of $\partial K \cap \partial L$. Here a component is non-trivial if it contains an edge.

DEFINITIONS. Let $\mu \in \text{Path}(M)$. Then μ is a *piece* if $\mu \in \text{Ov}(D, E)$ for some D, E in Reg(M) which satisfy $CN(1)$ (see [4]). Thus μ is a piece in M if $\phi(\mu)$ is a piece in the sense of 1.4.1. Denote by $\mathscr{P} = \mathscr{P}(M)$ the set of all the pieces of M. For a natural number k let

$$\mathscr{P}^k = \{\mu_1 \cdots \mu_k \mid \mu_i \in \mathscr{P}(M),\ \mu_1 \cdots \mu_k \in \text{Path}(M)\}.$$

More generally, if $S_1, \ldots, S_r$ are subsets of $\mathscr{P}$ then

$$S_1 \cdots S_r = \{\mu \in \mathrm{Path}(M) \mid \mu = \mu_1 \cdots \mu_r, \mu_i \in S_i, i = 1, \ldots, r\}.$$

For $D \in \mathrm{Boreg}(M)$ let $\mathrm{out}(D) = \mathrm{Ov}(D, D_\infty)$ and let $\mathrm{in}(D)$ be the complement of $\mathrm{out}(D)$ in ∂D. Thus $\mathrm{in}(D) \cdot \mathrm{out}(D)$ is a boundary cycle of D. Let

$$\mathscr{P}_{n-1} = \{\mu \in \mathrm{Ov}(D, E) \mid D, E \in \mathrm{Reg}(M), \phi(\mu) \in P_{n-1}\}.$$

Here P_{n-1} is defined as in 1.5.4. Denote $A = 2$, and $B = \beta$.

2.1.2. Let M be an $\mathscr{R}$-diagram and let $D \in \mathrm{Reg}(M)$. Assume that D satisfies CN(1) (see [4]). Since $\phi(\partial D) \in \mathscr{R}$, D has a boundary cycle $\omega = \omega(D)$ which can be decomposed by $\omega(D) = \omega_1\beta_1\omega_2\gamma_1\omega_3\beta_2\omega_4\gamma_2$ such that $\phi(\omega_i) = W_{n-2}^{(-1)^i}$, $\phi(\beta_i) = B^{(-1)^{i+1}}$ and $\phi(\gamma_i) = B^{(-1)^i}$ if $\phi(\omega(D)) = R$ and $\phi(\gamma_i) = B^{(-1)^{i+1}}$ if $\phi(\omega(D)) = R^{-1}$ (see 1.1.1). By the Uniqueness Theorem (1.5.2) this is the only decomposition of $\omega(D)$ to subpaths with the same labeling as above. Therefore we may uniquely define boundary paths $\omega_i(D)$, $i = 1, 2, 3, 4$, $\beta_i(D)$, $\gamma_i(D)$, $i = 1, 2$.

For a submap N of M introduce the following notation:

$$\Omega(N) = \{\omega_i(D) \mid D \in \mathrm{Reg}(N)\},$$

$$\Omega^\varepsilon(N) = \{\mu \in \Omega(N) \mid \phi(\mu) = W_{n-2}^\varepsilon\}, \varepsilon = 1, -1, \text{ and}$$

$$\hat{\Omega}^\varepsilon(N) = \bigcup_{k=0}^{\infty} (\Omega^\varepsilon(N))^k;$$

$$\Gamma(N) = \{\gamma_i(D) \mid D \in \mathrm{Reg}(N)\}, \hat{\Gamma}(N) = \bigcup_{k=0}^{\infty} (\Gamma(N))^k;$$

$$\Lambda(N) = \{\beta_i(D) \mid D \in \mathrm{Reg}(N)\}, \hat{\Lambda}(N) = \bigcup_{k=0}^{\infty} (\Lambda(N))^k.$$

Note that the sign of $\phi(\omega_i)$ changes from $+$ to $-$ when ω_i is followed by γ_j, and the sign of $\phi(\omega_i)$ changes from $-$ to $+$ when ω_i is followed by β_j.

2.1.3. Let $\mu \in U_1 \cdots U_r$, $U_i \in \{\Omega, \Lambda, \Gamma\}$. Denote by $\sigma_X(\mu)$ the sum of the exponents of $\phi(U_j)$, where $U_j \in X$. Here $X \in \{\Omega, \Lambda, \Gamma\}$. Thus if $\mu = \mu_1\mu_2\mu_3\mu_4$ where $\mu_1 \in \Omega$, $\mu_2 \in \Gamma$, $\mu_3 \in \Omega$, $\mu_4 \in \Lambda$ and $\phi(\mu_1) = W_{n-2}$, $\phi(\mu_3) = W_{n-2}^{-1}$, $\phi(\mu_2) = B$, $\phi(\mu_4) = B^{-1}$ then $\sigma_\Gamma(\mu) = 1$, $\sigma_\Lambda(\mu) = -1$ and $\sigma_\Omega(\mu) = 0$. Note that σ is well defined due to the uniqueness theorem. Note also that if ω is a boundary path of a region D of M, as defined in 2.1.2, then $\sigma_X(\omega) = 0$ for every $X \in \{\Omega, \Lambda, \Gamma\}$.

2.1.4. (a) Let M be an $\mathscr{R}$-diagram over F. Denote by $\mathscr{M}_F$ the set of all $\mathscr{R}$-diagrams which can be obtained from M by carrying out a sequence of diamond moves on M (see [1, §4, p. 160]). Diamond moves never change the boundary of M, the labels on it, nor change the number of regions, but only the way they are arranged in the diagram.

(b) If $M' \in \mathscr{M}$ we shall denote by D' the region of M' obtained from $D \in \mathrm{Reg}(M)$ by the sequence of diamond moves which leads to M'. If $\mu \in \mathrm{Ov}(D_1, D_2)$, D_1, $D_2 \in \mathrm{Reg}(M)$, let $\mu' = \mathrm{Ov}(D_1', D_2')$ be the corresponding path. Similarly, if S is a submap of M, then S' will stand for the corresponding submap in M'.

(c) Let $\hat{\mathscr{P}}_{n-1} = \{\mu \in \mathscr{P}(M) \mid \mu' \in \mathscr{P}_{n-1} \text{ for some } M' \in \mathscr{M}\}$.

2.1.5. Definitions. (a) Let $D_1, D_2 \in \mathrm{Reg}(M)$. D_1 and D_2 are *friends in M* if $\mathrm{Ov}(D_1, D_2) \cap \hat{\mathscr{P}}_{n-1} \neq \varnothing$.

(b) Let $D, E \in \mathrm{Reg}(M)$. D and E are *linked in M* if there exists a sequence $D_0, \ldots, D_t$ of regions of M with $D_0 = D$ and $D_t = E$ such that D_i and D_{i+1} are friends in M, $0 \leqq i \leqq t-1$.

Clearly, linkedness is an equivalence relation. Denote by $\Delta_0(D)$ the set of the regions in M which are linked to D. Thus $\Delta_0(D)$ is the equivalence class which contains D. Define $\Delta(D) = \mathrm{Int}\,(\overline{\bigcup}_{E \in \Delta_0(D)} E)$. Here $\mathrm{Int}(X)$ is the interior of X, $X \subseteq \mathscr{E}^2$, and $\bar{X}$ is the closure of X in $\mathscr{E}^2$.

The following definition is a special case of a construction due to E. Rips (see [10]).

2.1.6. Definition. (The derived map) Let M be a simply connected $\mathscr{R}$-diagram with connected interior. Assume that for each $D \in \mathrm{Reg}(M)$, $\Delta(D)$ is simply connected. Then we may consider the diagram M^d which is the regular diagram with $\mathrm{Reg}(M^d) = \{\Delta(D) \mid D \in \mathrm{Reg}(M)\}$. We call M^d the *derived diagram*. Obviously, $\partial(M^d) = \partial M$.

2.2. Theorem. *Let M be simply connected.*

(a) *For every $D \in \mathrm{Reg}(M)$, $\overline{\Delta(D)}$ is connected and simply connected. Moreover $\Delta(D)$ satisfies $C(4)$ & $T(4)$ as a map over R and it has a boundary label over $\langle W_{n-3}, B \rangle$.*

(b) *Let $\Delta_1, \Delta_2 \in \mathrm{Reg}(M^d)$ and assume that $\mu \in \mathrm{Ov}(\Delta_1, \Delta_2)$. Then $\phi(\mu)$ is a subword of $W_{n-3}^{\varepsilon} B^k W_{n-3}^{-\varepsilon}$, $\varepsilon \in \{1, -1\}$, $k \in \mathbf{Z}$.*

(c) *M^d satisfies the condition $C(8)$.*

(d) *Let $\Delta \in \mathrm{Reg}(M^d)$ and let $\mu \in \mathscr{P}^3(M^d)$ be a boundary path of Δ. Then*

$|\rho(\phi(\mu))| < \frac{1}{2}|\rho(\partial(\Delta))|$. *Moreover* W^{ε}_{n-3} *is a label of a boundary path* v *of* Δ, *such that* $v \cap \mu = \emptyset$. *Here* $\varepsilon \in \{1, -1\}$.

We prove the theorem by induction on $|M|$. For $|M| = 1$ the theorem is clear. So we let $|M| \geqq 2$ and assume the following hypothesis:

($\mathscr{H}$) *Theorem* 2.2 *holds for every* $\mathscr{R}$*-diagram* K *with* $|K| < |M|$ *that satisfies the conditions of the Theorem. In particular* ($\mathscr{H}$) *implies that* K *satisfies* $CN(2)$ (see [4]).

2.3. We begin the proof of Theorem 2.2(a). To this end we introduce below a special kind of submap contained in each $\Delta(D)$. These submaps, by definition, satisfy part (a) of Theorem 2.2. We show in 2.4.2 that they coincide with $\Delta(D)$. This clearly proves the first part of Theorem 2.2(a).

DEFINITION. Let M be a simply connected or annular $\mathscr{R}$-diagram with connected interior and let M_0 be a regular simply connected or annular subdiagram of M with connected and simply connected interior. M_0 is *standard* if it satisfies the following conditions:

(1) If M_0 is annular then M is annular and ∂M_0 is homotopic to ∂M.

(2) Let $D_1, D_2 \in \hat{\mathrm{R}}\mathrm{eg}(M_0)$. Then $\mathrm{Ov}(D_1, D_2) \subseteq \hat{\mathscr{P}}_{n-1}$ and $\phi(\mu) \in \langle W_{n-2}, B\rangle$ for every $\mu \in \mathrm{Ov}(D_1, D_2)$.

(3) For every connected component v of ∂M_0, there is a $D \in \mathrm{Boreg}(M_0)$ and $\mu \in \mathrm{Ov}(D, v)$ such that the boundary cycle θ of M_0 defined by $o(\theta) = t(\theta) = o(\mu)$ satisfies the following:

(i) $\theta \in (\hat{\Omega}^{-1}\hat{\Lambda}\hat{\Omega}\hat{\Gamma})^k$ for some $k \in \mathrm{N}$;

(ii) $\sigma_\Omega(\theta) = \sigma_\Lambda(\theta) = \sigma_\Gamma(\theta) = 0$.

REMARKS. (1) Observe that $M_0 = D$, for each $D \in \mathrm{Reg}(M)$ is standard.

(2) We may consider M_0 as a diagram over $\langle W_{n-2}, B\rangle$.

(3) Considering the decomposition of θ in part (3) of the definition, observe that a component of θ in $\hat{\Gamma}$ is preceded by a component in $\hat{\Omega}$ and followed by a component in $\hat{\Omega}^{-1}$, while a component of θ in $\hat{\Lambda}$ is preceded by a component from $\hat{\Omega}^{-1}$ and followed by a component from $\hat{\Omega}$. Consequently, if S_1 and S_2 are two standard diagrams and μ is a connected subpath of $\partial S_1 \cap \partial S_2$ such that $\mu \in \Lambda(S_1)\Omega(S_1)$, then necessarily $\mu \in \Lambda(S_2)\Omega(S_2)$ (i.e., $\mu \notin \Gamma(S_2)\Omega(S_2)$). More generally, if $\mu \in X(S_1)Y(S_1)$ then $\mu \in X(S_2)Y(S_2)$ where $X, Y \in \{\Lambda, \Omega, \Gamma\}$. Using diamond moves on the regions containing subpaths of μ on their boundary, the last observation implies that if $\mu = \xi\eta$, $\xi \in X(S_1)$, $\eta \in Y(S_1)$, $X \neq Y$, where X, $Y \in \{\Lambda, \Omega, \Gamma\}$ then ξ, $\eta \in \hat{\mathscr{P}}_{n-1}$. For if $\mu = \lambda\omega$, $\lambda \in \Lambda(S_1)$, $\omega \in \Omega(S_1)$ and

$D \in \mathrm{Boreg}(S_1)$ such that $\lambda \subseteq \partial D$, then λ is followed by a member ω_1 of $\Omega(D)$, hence we may choose an $M' \in \mathcal{M}$ such that $\lambda' \omega_1' = \mu'$, showing that λ, $\omega_1 \in \hat{\mathscr{P}}_{n-1}$. The same argument applies to the general case $\mu \in X(S_1)Y(S_1)$, $X \neq Y$.

2.4.1. The following lemma is an immediate consequence of definition 2.3(2). We omit its proof.

LEMMA. *Let S be a standard subdiagram of the simply connected diagram M and let $E \in \mathrm{Reg}(M \setminus S)$. Let $\mu_0 \in \mathrm{Ov}(E, S)$ and let μ be a subpath of μ_0. Assume that $\mu = \mu_1\mu_2$ such that $\mu_1 = \partial D_1 \cap \mu$ and $\mu_2 = \partial D_2 \cap \mu$, D_1, D_2 in* $\mathrm{Boreg}(S)$ *and $\mu_1 \neq 0$, $\mu_2 \neq 0$. If $\mu_1 \in \hat{\mathscr{P}}_{n-1}$ then $\mu_2 \in \hat{\mathscr{P}}_{n-1}$. More generally, if $\mu = \mu_1 \cdots \mu_n$ and $\mu_i \in \hat{\mathscr{P}}_{n-1}$ for some i, then $\mu_i \in \hat{\mathscr{P}}_{n-1}$ for every i, $i = 1, \ldots, n$.*

2.4.2. Let M be a simply connected $\mathscr{R}$-diagram with a connected interior and let $D \in \mathrm{Reg}(M)$. Let S' be a standard submap of $M' \in \mathcal{M}$ containing D' such that $|S'|$ is maximal possible when M' ranges over all $\mathcal{M}$. Let $\Delta' = \Delta(D')$. Then obviously $S' \subseteq \Delta$. We claim

PROPOSITION. $S' = \Delta'$. *In particular Δ' is simply connected and standard, hence Δ is simply connected.*

Clearly this proposition proves part (a) of Theorem 2.2 for the simply connected case.

PROOF OF THE PROPOSITION. It is enough to prove that $S' = \Delta'$. Assume by way of contradiction that $S' \neq \Delta'$. Then since Δ' has connected interior, there is a region $E \in \Delta' \setminus S'$ such that $\mathrm{Ov}(E, S') \cap \hat{\mathscr{P}}_{n-1} \neq \varnothing$. Consider $S' \cup \{E\}$. There are two cases.

Case 1. $\mathrm{Ov}(E, S') = \{\mu\}$.

Then $S' \cup \{E\}$ is simply connected. Notice that if $\phi(\mu) \in \langle W_{n-2}, B \rangle$ then all parts of Definition 2.3 hold for $S' \cup \{E\}$. But since $\phi(\mu) \in \hat{P}_{n-1}$, applying a sequence of diamond moves on certain regions in $M' \setminus S'$, we get a diagram M'' in which $S'' = S'$ (since we did not change regions of S') and $\phi(\mu) \in \langle W_{n-2}, B \rangle$. Thus $S' \cup \{E\}$ is a standard diagram, violating the maximality of S' in Δ'.

Case 2. $\partial E \cap \partial S'$ is not connected.

Then $S' \cup \{E\}$ is not simply connected. Let H be a simply connected component of $M' \setminus (S' \cup \{E\})$ with connected interior. Then as $|H| < |M|$, $\mathscr{H}$ of 2.2 applies to H, hence H^d has a corner region P by part (c) of Theorem 2.2 and Theorem A in [4]. (See [4, p. 85] for the definition of corner regions.)

Let $\mu = \partial P \cup \partial S'$. By [4, 5] μ is connected. Thus if we show that there is a region $D \in \mathrm{Boreg}(P)$ such that $\mathrm{Ov}(D, S') \in \hat{\mathscr{P}}_{n-1}$ then $S' \cup \{D\}$ is standard, leading to a contradiction to the maximality of S', as in Case 1. Following this idea, observe that by part (d) of Theorem 2.2 and part c(iii) of Definition 2.3 applied to H, μ must contain a subpath μ_0 such that

$$\mu_0 \in (\Omega(S')\Gamma(S')) \cup (\Gamma(S')\Omega^{-1}(S')) \cup (\Omega^{-1}(S')\Lambda(S')) \cup (\Lambda(S')\Omega(S')).$$

But then it follows from Remark 2.3(3) that $\mu_0 \in \hat{\mathscr{P}}^2(P) \cap \hat{\mathscr{P}}_{n-1}(P)$, showing the existence of a $D \in \mathrm{Boreg}\,(P)$ with $\mathrm{Ov}(D, S') \in \hat{\mathscr{P}}_{n-1}$. This contradiction completes the proof of the proposition and the first part of Theorem 2.2(a). The "moreover" part follows immediately from the last proposition.

2.4.3. We turn now to the proof of the rest of Theorem 2.2.

Proof of 2.2(b).

Case 1. $\Lambda(\Delta_i) \cap \mu = \Gamma(\Delta_i)$, $\mu = \varnothing$ for $i = 1, 2$. Then μ is a subpath of a member of Ω, hence $\phi(\mu)$ is a subword of an element $u \in P_{n-1}$.

Case 2. $X(\Delta_i) \cap \mu \neq \varnothing$ for some $X \in \{\Lambda, \Gamma\}$ and $i \in \{1, 2\}$. Thus, by Theorem 1.5.3, $\phi(\mu)$ is a subword of a power of an element $u \in P_{n-2}$, i.e.,

$$u_0\rho(\phi(u))u_1 = W_{n-3}^{\varepsilon} B^k W_{n-3}^{-\varepsilon} \qquad \text{where } \varepsilon \in \{1, -1\}, \quad k \in \mathbf{Z}.$$

This completes the proof of 2.2(b).

Proof of 2.2(c). Immediate by 2.2(b) and Definitions 2.3(3)(i) and (ii).

Proof of 2.2(d). Immediate by 2.2(b) and Definitions 2.3(3)(i) and (ii).

The proof of Theorem 2.2 is complete.

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